MODELING AND NUMERICAL SIMULATION OF A GRAND PIANO.

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Talk Abstract

We consider a complete model of a piano which accounts for the acoustical behavior of the instrument from excitation to soundand, and we propose a numerical discretisation. The model is described as well as the numerical methods used for its discretisation. Nonlinearities and couplings are treated in such a way that energy techniques ensure numerical stability. Numerical results are presented and compared to measurements.

Introduction

After a key has been engaged, a nonlinear hammer strikes either one, two or three strings. The percussive timbre of the piano is attributed to the presence of a longitudinal vibration in the string, which is nonlinearly coupled to the transversal vibration thanks to a geometrically exact description of the string (see [1]). The transversal and longitudinal vibrations of the strings are transmitted to the structure through the bridge, thanks to a nonstandard coupling condition. A Reissner Mindlin plate model is used to describe the soundboard, which radiates the sound in the air. All the couplings of the continuous system are reciprocal so that the global energy is preserved, or decaying if physical dissipation is introduced.

A numerical discretization is proposed for the whole system. A first difficulty is due to the nonlinearity of both strings and hammer. Another one arises from the reciprocal couplings of the system: hammer / strings, strings / soundboard, soundboard / air. Numerical stability is achieved through an energy technique: each sub-system either conserves a discrete energy, or transmits it to another sub-system, so that the resulting complete numerical scheme conserves a discrete and consistent global energy, or reproduces the physical dissipation decay.

Higher order finite elements are used for space discretisation in 1D (on the string), 2D (on the soundboard) and 3D (in the air). Time discretisation is the main issue, especially on the string and the soundboard where it is tackled differently:

- An innovating, energy preserving scheme is built for the nonlinear system of string equations,
- A piecewise analytic resolution in time is done on the soundboard, thanks to a modal approach in space.

1 String system

1.1 Nonlinear stiff string equations

Considering the longitudinal vibration of the string is a crucial point to reproduce the percussive timbre of the piano. Moreover, dispersion plays a great role in the timbre of musical instruments. This is why modeling the stiffness of the string is also very important in our case. To do so, we consider a prestressed nonlinear version of the Timoshenko beam model. We consider an infinitely thin string, parametrized at rest with $x \in [0, L]$, where L is the length of the string. The unknowns of the model are $u(x,t)$, the transversal displacement of the string, $v(x,t)$, its longitudinal displacement and $\varphi(x,t)$, the deviation of the cross-sections from the normal of the string. The unknowns u and v are coupled via the so-called geometrically exact model [2] which is nonlinear because it accounts for large deformations. We will call ρ the volumic mass of the string, A the area of its section, E its Young's modulus, T_0 its tension at rest, I the stiffness inertia coefficient of the string, G its shear coefficient, and k' the Timoshenko parameter. The nonlinear stiff string model reads:

$$\partial_t^2 M\mathbf{q} - \partial_x\big(A\partial_x\mathbf{q} + B\mathbf{q} + \nabla H(\partial_x\mathbf{q})\big) + {}^tB\partial_x\mathbf{q} + C\mathbf{q} = 0 \tag{1}$$

where the coefficient matrices are defined by

$$M = \begin{pmatrix} \rho A & 0 & 0 \\ 0 & \rho A & 0 \\ 0 & 0 & \rho I \end{pmatrix}, \quad A = \begin{pmatrix} T_0 + AGk' & 0 & 0 \\ 0 & EA & 0 \\ 0 & 0 & EI \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & AGk' \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & -AGk' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $H : \mathbb{R}^N \mapsto \mathbb{R}$, with $N = 3$ and $\mathbf{q} = (u, v, \varphi)$,

$$H(\mathbf{q}) = (EA - T_0)\big[u^2 + (1+v) - \sqrt{u^2 + (1+v)^2}\big].$$

Considering Dirichlet boundary conditions for u and v and Neumann for φ, the system (1) preserves the energy

$$\mathcal{E}_s(t) = \frac{1}{2}\big\|\partial_t\mathbf{q}\big\|_M^2 + \frac{1}{2}\big\|\partial_x\mathbf{q}\big\|_A^2 + \frac{1}{2}\big\|\mathbf{q}\big\|_C^2 + \frac{1}{2}\langle\partial_x\mathbf{q}, \mathbf{q}\rangle_B + \int_0^L H(\partial_x\mathbf{q}) \tag{2}$$

where for any vector $\mathbf{u}$ and matrix A,

$$\|\mathbf{u}\|_A^2 = \int_0^L A\mathbf{u}\cdot\mathbf{u}, \text{ and } \langle \partial_x \mathbf{u}, \mathbf{u}\rangle_A = \int_0^L A\,\partial_x\mathbf{u}\cdot\mathbf{u}$$

This energy is positive since:

$$\begin{cases} A = A_S + A_T, \text{ with } A_S = \mathrm{diag}(T_0, EA, 0) & \text{(3a)} \\ \begin{pmatrix} C & {}^tB \\ B & A_T \end{pmatrix} \text{ is a positive matrix,} & \text{(3b)} \\ \frac{1}{2}\, A_S\,\mathbf{q}\cdot\mathbf{q} + H(\mathbf{q}) \geq 0, \qquad \forall\,\mathbf{q}\in\mathbb{R}^3. & \text{(3c)} \end{cases}$$

1.2 *Numerical approximation*

Space discretisation of (1) is done with higher order finite elements. For any $\tilde{\mathbf{q}}$ in the finite elements space $\mathcal{V}_h$,

$$\frac{d^2}{dt^2}\int_0^L M\,\mathbf{q}\cdot\tilde{\mathbf{q}} + \int_0^L \big(A\,\partial_x\mathbf{q} + B\,\mathbf{q} + \nabla H(\partial_x\mathbf{q})\big)\cdot\partial_x\tilde{\mathbf{q}} + \int_0^L {}^tB\,\partial_x\mathbf{q}\cdot\tilde{\mathbf{q}} + \int_0^L C\,\mathbf{q}\cdot\tilde{\mathbf{q}} = 0.$$

For time discretisation, we have chosen to handle differently the linear and nonlinear parts of the system. For the linear part, we use a θ scheme. Choosing $\theta = 1/12$ provides fourth order time accuracy and reduces numerical dispersion. We call $\mathbf{Q}\in\mathbb{R}^M$ ($M = \dim(\mathcal{V}_h)$) the vector of coordinates of $\mathbf{q}$ in a finite elements basis of the space $\mathcal{V}_h$ and we denote, $\forall\,\mathbf{Q}^n\in\mathbb{R}^M$, $n\geq 0$:

$$\begin{cases} [\mathbf{Q}]^n_{\Delta t^2} = \dfrac{\mathbf{Q}^{n+1} - 2\,\mathbf{Q}^n + \mathbf{Q}^{n-1}}{\Delta t^2} \\ \{\mathbf{Q}\}^n_\theta = \theta\,\mathbf{Q}^{n+1} + (1-2\theta)\,\mathbf{Q}^n + \theta\,\mathbf{Q}^{n-1} \end{cases}$$

For the nonlinear part, we use the scheme proposed in [3] which preserves a discrete energy. Given $k\in[1,N]$ and $\sigma:\Sigma_k\mapsto\{-1,1\}$, $\Sigma_k = [1,N]\setminus k$, we denote:

$$\delta_k^\sigma H(\mathbf{q}^{n+1},\mathbf{q}^{n-1}) = \frac{H(q_k^{n+1}, q_{\ell\neq k}^{n+\sigma(\ell)}) - H(q_k^{n-1}, q_{\ell\neq k}^{n+\sigma(\ell)})}{q_k^{n+1} - q_k^{n-1}}$$

For all $(\mathbf{Q}^+,\mathbf{Q}^-,\tilde{\mathbf{Q}})\in\mathbb{R}^M$, we define

$$\begin{cases} \overline{\nabla}^\sigma H(\mathbf{Q}^+,\mathbf{Q}^-)\cdot\tilde{\mathbf{Q}} = \displaystyle\int_0^L \mathrm{Vect}\Big\{\delta_k^\sigma H(\partial_x\mathbf{q}^+,\partial_x\mathbf{q}^-)\Big\}\cdot\partial_x\tilde{\mathbf{q}} \\ H(\mathbf{Q}) = H(\partial_x\mathbf{q}) \end{cases}$$

and A_h, B_h, C_h, the classical finite elements matrices. Finally we consider the weighting coefficients $\zeta(\sigma)$ introduced in [3], which satisfy $\sum\zeta(\sigma) = 1$ and we define:

$$\overline{\nabla} H(\mathbf{Q}^+,\mathbf{Q}^-) = \sum_{\sigma\in\Sigma_k}\zeta(\sigma)\,\overline{\nabla}^\sigma H(\mathbf{Q}^+,\mathbf{Q}^-).$$

The numerical scheme that we consider is the following:

$$M_h[\mathbf{Q}]^n_{\Delta t^2} + A_h\{\mathbf{Q}\}^n_\theta + B_h\{\mathbf{Q}\}^n_\theta + \overline{\nabla} H(\mathbf{Q}^{n+1},\mathbf{Q}^{n-1}) + {}^tB_h\{\mathbf{Q}\}^n_\theta + C_h\{\mathbf{Q}\}^n_\theta = 0 \quad (4)$$

which preserves a discrete energy. Introducing

$$A_h = A_{S,h} + A_{T,h} \quad \text{and} \quad K_{T,h} = A_{T,h} + B_h + {}^tB_h + C_h,$$

the discrete energy, consistent with $\mathcal{E}_s(t)$, reads:

$$\begin{aligned} \mathcal{E}_s^{n+\frac{1}{2}} = {} & \frac{1}{2}\Big\|\frac{\mathbf{Q}^{n+1}-\mathbf{Q}^n}{\Delta t}\Big\|^2_{M_h} + \frac{1}{2}\Big\|\frac{\mathbf{Q}^{n+1}+\mathbf{Q}^n}{2}\Big\|^2_{K_{T,h}} \\ & + \frac{1}{4}\Big[\big\|\mathbf{Q}^{n+1}\big\|^2_{A_{S,h}} + \big\|\mathbf{Q}^n\big\|^2_{A_{S,h}}\Big] \\ & + \int_0^L \frac{H(\mathbf{Q}^{n+1}) + H(\mathbf{Q}^n)}{2} \\ & + \frac{\Delta t^2}{2}\Big[(\theta-\frac{1}{2})\Big\|\frac{\mathbf{Q}^{n+1}-\mathbf{Q}^n}{\Delta t}\Big\|^2_{A_{S,h}} \\ & + (\theta-\frac{1}{4})\Big\|\frac{\mathbf{Q}^{n+1}-\mathbf{Q}^n}{\Delta t}\Big\|^2_{K_{T,h}}\Big] \end{aligned}$$

The positivity of this energy leads to the numerical stability of the numerical scheme. The discrete equivalent of conditions (3) is naturally fulfilled with finite elements methods. Hence, numerical stability is achieved when

$$M_h + (\theta-1/2)\Delta t^2 A_{S,h} + (\theta-1/4)\Delta t^2(A_{T,h} + B_h + {}^tB_h + C_h)$$

is semi-definite positive. This requirement leads to a CFL condition on Δt as soon as $\theta < 1/2$.

2 Hammer / strings interaction

2.1 *Coupling equations*

The interaction with the hammer is essential for timbre quality and realism in sound synthesis. We will consider a contact with nonlinear interaction and hysteresis. The reality and geometry of the piano leads us to take into account the coupling of several (N_c) strings with only one hammer. We call $\mathbf{q}_i = (u_i, v_i, \varphi_i)$ the triplet of unknowns of the ith string. As the strings are slightly detuned (their tension at rest, T_0, is different, see [4]), each string has distinct H_i and A_i in (1).

The hammer's center of gravity is supposed to be moving along a straight line orthogonal to the string at rest. Its position is located by a scalar unknown $\xi(t)$. The parameters characterizing the mechanical behavior of the hammer are its mass M^{ham}, stiffness K^{ham} and dissipation R^{ham}, and the function Φ which links the force of interaction to the crushing of the hammer. The contact is distributed along the string through a repartition function δ^{ham} centered at the impact point x^{ham}, and we denote $\langle u_i \rangle = \int_0^L \delta^{\text{ham}}(x) u_i(x) dx$ the value of u_i averaged by δ^{ham}. We call $e_i(t) = \langle u_i \rangle(t) - \xi(t)$ the distance between the ith string and the hammer. The hammer's force is orthogonal to the string, hence is a right-hand side for the transversal motion. The coupled system can now be written (setting $\mathbf{e}_u = (1,0,0)$)

$$\begin{cases} M^{\text{ham}} \dfrac{d^2\xi}{dt^2}(t) = -\sum_i F_i^{\text{ham}}(t) & \text{(5a)} \\ F_i^{\text{ham}}(t) = K^{\text{ham}} \Phi\big(e_i(t)\big) + R^{\text{ham}} \dfrac{d}{dt}\Phi\big(e_i(t)\big) & \text{(5b)} \\ \partial_t^2 M \mathbf{q}_i - \partial_x(A_i \partial_x \mathbf{q}_i + B\mathbf{q}_i \nabla H_i(\partial_x \mathbf{q}_i)) \\ \qquad + {}^t B \partial_x \mathbf{q}_i + C\mathbf{q}_i = F_i^{\text{ham}}(t)\, \delta^{\text{ham}}(x)\, \mathbf{e}_u & \text{(5c)} \end{cases}$$

Setting $\Psi \geq 0$ such that $\Psi' = -\Phi$, the previous system respects the physical energy decay, with the energy:

$$\mathcal{E}_{h,s}(t) = \sum_{i=1}^{N_c} \Big[\mathcal{E}_{s,i}(t) + K^{\text{ham}} \Psi\big(e_i(t)\big)\Big] + \frac{M^{\text{ham}}}{2} \big|\xi'(t)\big|^2,$$

where $\mathcal{E}_{s,i}(t)$ is the energy (2) of the ith string ($\mathbf{q} \equiv \mathbf{q}_i$).

2.2 *Numerical approximation*

An energy preserving numerical approximation of (5) can be obtained by using the previous string's scheme (4) for each string and considering a leap frog scheme for the hammer. The contributions coming from the interaction must be discretized so that a total energy is preserved. Since the function Φ is nonlinear, we have to treat the hammer implicitly with the string, which is not a great over cost since it is a scalar unknown. Using the discrete version of the right-hand side $F_i^{\text{ham}}(t)$:

$$K^{\text{ham}} \frac{\Psi\big(e_i^{n+1}\big) - \Psi\big(e_i^{n-1}\big)}{e_i^{n+1} - e_i^{n-1}} - R^{\text{ham}} \frac{\Phi\big(e_i^{n+1}\big) - \Phi\big(e_i^{n-1}\big)}{2\Delta t},$$

the scheme respects the physical energy decay, with:

$$\mathcal{E}_{h,s}^{n+\frac{1}{2}} = \sum_{i=1}^{N_c} \Big[\mathcal{E}_{s,i}^{n+\frac{1}{2}} + K^{\text{ham}} \frac{\Psi\big(e_i^{n+1}\big) + \Psi\big(e_i^{n}\big)}{2}\Big] + \frac{M^{\text{ham}}}{2} \left|\frac{\xi^{n+1} - \xi^n}{\Delta t}\right|^2$$

3 Soundboard and vibroacoustics

3.1 *Vibroacoustics equations*

The piano soundboard is a complex structure which can be seen as a thick plate with ribs and bridges. We use Reissner-Mindlin model which considers the plate transversal displacement u_p and two deflection angles $(\theta_{1,p}, \theta_{2,p}) = \underline{\theta}_p$. Moreover, the plate ω interacts with the air Ω through the coupling conditions of vibroacoustics. The most natural way of deriving it from physics is to write the system in the velocity / pressure form. Here, for numerical purposes, we have chosen to introduce the time primitive of the pressure, P, as unknown in the air, so that the equations are:

$$\begin{cases} \begin{cases} c_\theta \dfrac{\partial^2 \underline{\theta}_p}{\partial t^2} + A\, \underline{\theta}_p + C\, u_p = 0 \\ c_u \dfrac{\partial^2 u_p}{\partial t^2} + B\, u_p + {}^t C\, \underline{\theta}_p = f\, \chi_\omega(x,y) + [\dfrac{\partial P}{\partial t}] \end{cases} \\ \dfrac{1}{c^2} \dfrac{\partial^2 P}{\partial t^2} - \Delta P = 0 \qquad \text{in } \Omega \\ -\rho_f \dfrac{\partial P}{\partial n} = \dfrac{\partial u_p}{\partial t} \qquad \text{on } \omega \end{cases}$$

where, calling δ the thickness and ρ the volumic mass of the plate, ρ_f the volumic mass of the fluid and c the celerity of sound in the air,

$$c_\theta = \rho\, \delta^3/12, \quad c_u = \rho\, \delta, \quad B\, u = -\delta\, \text{div}(G\, \underline{\nabla} u),$$
$$A\, \underline{\theta} = -(\delta^3/12)\, \underline{\text{Div}}(\underline{\underline{C}}\, \underline{\varepsilon}(\underline{\theta})) + \delta\, G\, \underline{\theta}, \quad C\, u = \delta\, G\, \underline{\nabla} u$$

χ_ω is a repartition function for the loading f and $[X]$ denotes the jump of X through the plate interface.

If $f = 0$, this system preserves the energy

$$\begin{aligned} \mathcal{E}_{p,f}(t) = \frac{c_u}{2} \left\|\partial_t u_p\right\|^2 + \frac{c_\theta}{2} \left\|\partial_t \theta_p\right\|^2 + \frac{1}{2} \left\|\underline{\theta}_p\right\|_A^2 \\ + \frac{1}{2} \left\|u_p\right\|_B^2 + \frac{1}{2} \left\|C u_p + \underline{\theta}_p\right\|^2 \\ + \frac{1}{2\rho_f\, c^2} ||\partial_t P||_\Omega^2 + \frac{1}{2\rho_f} ||\nabla P||_\Omega^2 \end{aligned}$$

3.2 *Numerical approximation*

Higher order finite elements are performed on the plate and in the 3D volume. They lead to the following semi-discrete system on $\Lambda_h^{EF} = {}^t\big(U_{p,h}, \Theta_{p,h}\big)$ and P_h:

$$\begin{cases} \partial_t^2\, \mathbb{M}_h \Lambda_h^{EF} + \mathbb{R}_h \Lambda_h^{EF} = \begin{pmatrix} f J_h + {}^t \underline{\mathcal{C}}_h \partial_t P_h \\ 0 \end{pmatrix} & \text{(7a)} \\ \mathcal{M}_h \partial_t^2 P_h + \mathcal{K}_h P_h + {}^t \underline{\mathcal{C}}_h \partial_t U_{p,h} = 0 & \text{(7b)} \end{cases}$$

where $\mathbb{M}_h$ and $\mathcal{M}_h$ are the mass matrices of the plate and the acoustics equations, $\mathbb{R}_h$ and $\mathcal{R}_h$ the associated rigidity matrices and $\underline{\mathcal{C}}_h$ the "jump" matrix across ω.

We choose a different time discretisation on the plate and in the fluid. An analytic resolution in time is possible on the plate as follows. We diagonalize $\mathbb{R}_h$ in a $\mathbb{M}_h$-orthogonal basis : there exist $\mathbb{P}_h$ a change of basis matrix and $\mathbb{D}_h$ a diagonal matrix such that:

$$\begin{cases} {}^t\mathbb{P}_h\mathbb{R}_h\mathbb{P}_h = \mathbb{D}_h \\ {}^t\mathbb{P}_h\mathbb{M}_h\mathbb{P}_h = \mathbb{I}_d \end{cases}, \quad \begin{cases} \Lambda_h^{EF} & = \mathbb{P}_h\Lambda_h^{mod} \\ \Lambda_h^{mod} & = {}^t\mathbb{P}_h\mathbb{M}_h\Lambda_h^{EF} \end{cases}$$

Since the diagonalization transformes the plate system into decoupled EDOs, we can solve it analytically on a shifted time grid, provided that the right hand side is maintained constant along a time step $[t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}}]$. The discrete version of (7a) is:

$$\begin{cases} \partial_t^2\Lambda_h^{mod} + \mathbb{D}_h\Lambda_h^{mod} = F\,{}^t\mathbb{P}_h J_h + {}^t\mathbb{P}_h\underline{\mathcal{C}}_h\dfrac{P_h^{n+1} - P_h^{n-1}}{2\Delta t}, \\ \quad\Lambda_h^{mod}(t = t^{n-\frac{1}{2}}) = \quad\Lambda_h^{mod,n-\frac{1}{2}}, \\ \partial_t\Lambda_h^{mod}(t = t^{n-\frac{1}{2}}) = \partial_t\Lambda_h^{mod,n-\frac{1}{2}}, \end{cases}$$

Acoustic resolution is done with an explicit leap-frog scheme. The discrete version of (7b) is:

$$\mathcal{M}_h[P_h]^n_{\Delta t^2} + {}^t\underline{\mathcal{C}}_h\mathbb{P}_h\frac{\Lambda_h^{mod,n+\frac{1}{2}} - \Lambda_h^{mod,n-\frac{1}{2}}}{\Delta t} = 0.$$

The coupling terms have been written such that they cancel each other, so that a discrete energy is preserved:

$$\mathcal{E}_{p,f}^{n+\frac{1}{2}} = \frac{1}{2}\left\|\frac{P_h^{n+1} - P_h^n}{\Delta t}\right\|^2_{\mathcal{M}_h} - \frac{\Delta t^2}{8}\left\|\frac{P_h^{n+1} - P_h^n}{\Delta t}\right\|^2_{\mathcal{K}_h} + \frac{1}{2}\left\|\frac{P_h^{n+1} + P_h^n}{2}\right\|^2_{\mathcal{K}_h} + \frac{1}{2}\left\|\partial_t\Lambda_h^{mod,n+\frac{1}{2}}\right\|^2 + \frac{1}{2}\left\|\Lambda_h^{mod,n+\frac{1}{2}}\right\|^2_{\mathbb{D}_h}$$

The positivity of this quantity, equivalent to the stability of the numerical scheme, is acquired as soon as the matrix $\mathcal{M}_h - \frac{\Delta t^2}{4}\mathcal{K}_h$ is semi-definite positive.

4 Strings / Soundboard coupling at the bridge

How are the longitudinal vibrations of the strings transmitted to the soundboard ? This is due to the fact that at rest, the strings are not strictly parallel to the soundboard, but a slight angle α is present. We introduce as new unknowns the two components of the transmitted forces : $F_i^{\mathcal{P}}(t)$ ans $G_i^{\mathcal{P}}(t)$ between each string and the soundboard such that $f = -\sum_i F_i^{\mathcal{P}}(t)$ and we add to each string's right hand side the vector $F_i^{\mathcal{P}}(t)\,\underline{\nu} + G_i^{\mathcal{P}}(t)\,\underline{\nu}^{\perp}$, where $\underline{\nu} = \big(\cos(\alpha), \sin(\alpha), 0\big)$. The missing equations which allow to determinate $F_i^{\mathcal{P}}(t)$ and $G_i^{\mathcal{P}}(t)$, are the continuity equations of acoustical and mechanical velocities:

$$\dot{\mathbf{q}}_i(x = L, t)\cdot\underline{\nu} = \int_\omega \chi_\omega(x, y)\,\dot{u}_p(t), \quad \forall\, i \in [1, N_c],$$

which moreover guarantee a global energy decay for the whole system. A centered discrete version of these equations are written in the modal basis of the soundboard.

5 Complete coupled model

The complete model arising from the hammer / strings model coupled to the vibroacoustics model through the bridge, as well as its discrete version, both respect a total physical energy decay, with:

$$\mathcal{E}_{h,s,p,f}(t) = \mathcal{E}_{h,s}(t) + \mathcal{E}_{p,f}(t), \quad \mathcal{E}_{h,s,p,f}^{n+\frac{1}{2}} = \mathcal{E}_{h,s}^{n+\frac{1}{2}} + \mathcal{E}_{p,f}^{n+\frac{1}{2}}.$$

Numerically, we manage to decouple the resolution on each sub-system thanks to Lagrange multipliers and Schur complement techniques. The nonlinear resolution of the system strings / hammer / Lagrange multipliers is tackled with a Newton method at each time step.

A numerical experiment is lead for string C2 on a 3 meter-long grand piano. Figure 1 represents a snapshot after 10 ms of the soundboard's displacement u_p in the horizontal plane, and the pressure P in the two vertical slices. The string (which is not represented here) is attached to the soundboard at the crossing point of the two slices.

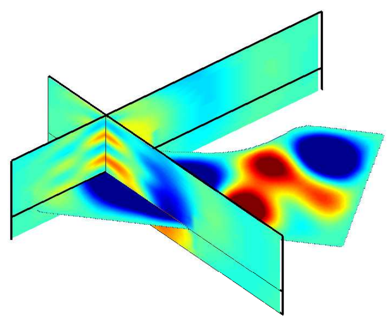

Figure 1: Sliced view of a snapshot.

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